PACKING 3-VERTEX PATHS IN 2-CONNECTED GRAPHS

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Abstract

Let $v(G)$ and $\lambda(G)$ be the number of vertices and the maximum number of disjoint 3-vertex paths in G, respectively. We give a construction that provides infinitely many 2-connected, cubic, bipartite, and planar graphs such that $\lambda(G) < \lfloor v(G)/3 \rfloor$.

Keywords: cubic, bipartite, planar, Λ-packing, Λ-factor.

1 Introduction

We consider undirected graphs with no loops and no parallel edges. All notions and facts on graphs, that are used but not described here, can be found in [1, 2, 12].

Given graphs G and H, an *H-packing* of G is a subgraph of G whose components are isomorphic to H. An *H-packing* P of G is called an *H-factor* if $V(P) = V(G)$. The *H-packing problem*, i.e. the problem of finding in G an H-packing, having the maximum number of vertices, turns out to be NP-hard if H is a connected graph with at least three vertices [3]. Let Λ denote a 3-vertex path. In particular, the Λ-packing problem is NP-hard. Moreover, this problem remains NP-hard even for cubic graphs [6].

Although the Λ-packing problem is NP-hard, i.e. possibly intractable in general, this problem turns out to be tractable for some natural classes of graphs (e.g. [4, 10]). It would be also interesting to find polynomial algorithms that would provide a good approximation solution for the problem. Below (see **1.1**, **1.2**, and **1.4**) are some examples of such results. In each case the corresponding packing problem is polynomially solvable.

Let $v(G)$ and $\lambda(G)$ denote the number of vertices and the maximum number of disjoint 3–vertex paths in G, respectively. Obviously $\lambda(G) \le \lfloor v(G)/3 \rfloor$.

In [5, 11] we answered the following natural question:

How many disjoint 3-vertex paths must a cubic n-vertex graph have?

1.1 *If G is a cubic graph, then $\lambda(G) \ge \lceil v(G)/4 \rceil$. Moreover, there is a polynomial time algorithm for finding a Λ-packing having at least $\lceil v(G)/4 \rceil$ components.*

Obviously if every component of G is K_4, then $\lambda(G) = v(G)/4$. Therefore the bound in **1.1** is sharp.

Let $\mathcal{G}_2^3$ denote the set of graphs with each vertex of degree at least 2 and at most 3.

In [5] we answered (among other results) the following question:

How many disjoint 3-vertex paths must an n-vertex graph from $\mathcal{G}_2^3$ have?

1.2 *Suppose that $G \in \mathcal{G}_2^3$ and G has no 5-vertex components. Then $\lambda(G) \ge \lceil v(G)/4 \rceil$.*

Obviously **1.1** follows from **1.2** because if G is a cubic graph, then $G \in \mathcal{G}_2^3$ and G has no 5-vertex components.

In [5] we also gave a construction that allowed to prove the following:

1.3 *There are infinitely many connected graphs for which the bound in* **1.2** *is attained. Moreover, there are infinitely many subdivisions of cubic 3-connected graphs for which the bound in* **1.2** *is attained.*

The next interesting question is:

How many disjoint 3-vertex paths must a cubic connected graph have?

In [7] we proved the following.

1.4 *Let $\mathcal{C}_n$ denote the set of connected cubic graphs with n vertices and $\lambda_n = \min\{\lambda(G)/v(G) : G \in \mathcal{C}_n\}$. Then for some $c > 0$,*

$$\frac{3}{11}\left(1 - \frac{c}{n}\right) \le \lambda_n \le \frac{3}{11}\left(1 - \frac{1}{n^2}\right).$$

The next natural question is:

1.5 Problem *How many disjoint 3-vertex paths must a cubic 2-connected graph have?*

This question is still open (namely, the sharp lower bound on the number of disjoint 3-vertex paths in a cubic 2-connected n-vertex graph is unknown).

There are infinitely many 2-connected and cubic graphs such that $\lambda(G) < \lfloor v(G)/3 \rfloor$.

As to cubic 3-connected graphs, an old open questions here is:

1.6 Problem *Is the following claim true:*
if G is a 3-connected and cubic graph, then $\lambda(G) = \lfloor v(G)/3 \rfloor$?

In [9] we discuss Problem **1.6** and show, in particular, that the claim in **1.6** is equivalent to some seemingly much stronger claims. Here are some results of this kind.

1.7 [9] *The following are equivalent for cubic 3-connected graphs G:*

(**z1**) $v(G) = 0 \bmod 6 \Rightarrow$ *G has a Λ-factor,*

(**z2**) $v(G) = 0 \bmod 6 \Rightarrow$ *for every $e \in E(G)$ there is a Λ-factor of G avoiding e,*

(**z3**) $v(G) = 0 \bmod 6 \Rightarrow$ *for every $e \in E(G)$ there is a Λ-factor of G containing e,*

(**z4**) $v(G) = 0 \bmod 6 \Rightarrow$ *$G - X$ has a Λ-factor for every $X \subseteq E(G)$, $|X| = 2$,*

(**z5**) $v(G) = 0 \bmod 6 \Rightarrow$ *$G - L$ has a Λ-factor for every 3-vertex path L in G,*

(**t2**) $v(G) = 2 \bmod 6 \Rightarrow$ *$G - \{x, y\}$ has a Λ-factor for every $xy \in E(G)$,*

(**f1**) $v(G) = 4 \bmod 6 \Rightarrow$ *$G - x$ has a Λ-factor for every $x \in V(G)$,*

(**f2**) $v(G) = 4 \bmod 6 \Rightarrow$ *$G - \{x, e\}$ has a Λ-factor for every $x \in V(G)$ and $e \in E(G)$.*

In [10] we have shown (in particular) that all claims in **1.7** except for ($z5$) are true for 3-connected claw-free graphs and ($z5$) is true for cubic, 2-connected, and claw-free graphs distinct from K_4 (see also [4]).

The problems similar to **1.6** are interesting for 2-connected and cubic graphs having some additional properties. For example,

1.8 Problem *Is $\lambda(G) = \lfloor v(G)/3 \rfloor$ true for every 2-connected, cubic, bipartite, and planar graph ?*

In this paper (see Section 3) we answer the question in **1.8** by giving a construction that provides infinitely many 2-connected, cubic, bipartite, and planar graphs such that $\lambda(G) < \lfloor v(G)/3 \rfloor$ (see also [8]).

2 Some notation, constructions, and simple observations

We consider undirected graphs with no loops and no parallel edges unless stated otherwise. As usual, $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively, and $v(G) = |V(G)$. If X is a vertex subset or a subgraph of G, then let $D(X, G)$ or simply $D(X)$, denotes the set of edges in G, having exactly one end–vertex in X, and let $d(X, G) = |D(X, G)|$. If $x \in V(G)$, then $D(x, G)$ is the set of edges in G incident to x, $d(x, G) = |D(x, G)|$, $N(x, G) = N(x)$ is the set of vertices in G adjacent to x, and $\Delta(G) = \max\{d(x, G) : x \in V(G)\}$. If $e = xy \in E(G)$, then let $End(e) = \{x, y\}$. Let $Cmp(G)$ denote the set of components of G and $cmp(G) = |Cmp(G)|$.

Let $\mathcal{C}(k)$ denote the set of cubic k-connected graphs, $k \in \{1, 2, 3\}$.

Let A and B be disjoint graphs, $a \in V(A)$, $b \in V(B)$, and $\sigma : N(a, A) \to N(b, B)$ be a bijection. Let $Aa\sigma bB$ denote the graph $(A - a) \cup (B - b) \cup \{x\sigma(x) : x \in N(a, A)\}$. We usually assume that $N(a, A) = \{a_1, a_2, a_3\}$, $N(b, B) = \{b_1, b_2, b_3\}$, and $\sigma(a_i) = b_i$ for $i \in \{1, 2, 3\}$ (see Fig. 1).

We also say that *$Aa\sigma bB$ is obtained from B by replacing vertex b by $(A - a)$ according to σ.*

Let B be a cubic graph and $X \subseteq V(B)$. Let $A(v)$, where $v \in X$, be a graph, a^v be a vertex of degree three in $A(v)$, and $A^v = A(v) - a^v$. By using the above operation, we

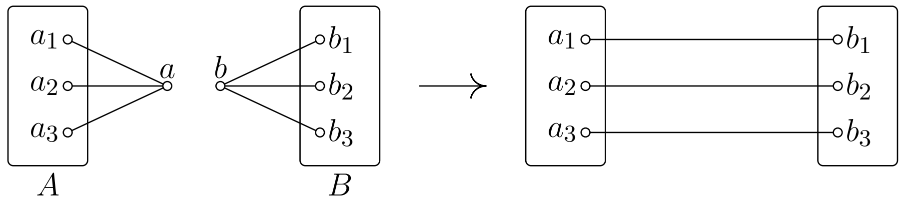

Figure 1: $Aa\sigma bB$

can build a graph $G = B\{(A(v), a^v) : v \in X\}$ by replacing each vertex v of B in X by A^v assuming that all $A(v)$'s are disjoint. Let $D^v = D(A^v, G)$. For each $u \in V(B) \setminus X$ let $A(u)$ be the graph having exactly two vertices u, a^u and exactly three parallel edges connecting u and a^u. Then $G = B\{(A(v), a^v) : v \in X\} = B\{(A(v), a^v) : v \in V(B)\}$. If, in particular, $X = V(G)$ and each $A(v)$ is a copy of K_4, then G is obtained from B by replacing each vertex by a triangle.

Let $E' = E(G) \setminus \cup\{E(A^v) : v \in V(B)\}$. Obviously, there is a unique bijection $\alpha : E(B) \to E'$ such that if $uv \in E(B)$, then $\alpha(uv)$ is an edge in G having one end-vertex in A^u and the other in A^v.

Let P be a Λ-packing in G. For $uv \in E(B)$, $u \neq v$, we write $u \neg^p v$ or simply, $u \neg v$, if P has a 3-vertex path L such that $\alpha(uv) \in E(L)$ and $|V(A^u) \cap V(L)| = 1$. Let P^v be the union of components of P that meet D^v in G.

Obviously

2.1 *Let k be an integer and $k \leq 3$. If A and B above are k-connected, cubic, bipartite, and planar graphs, then $Aa\sigma bB$ is also a k-connected, cubic, bipartite, and planar graph, respectively.*

From **2.1** we have:

2.2 *Let k be an integer and $k \leq 3$. If B and each A_v is a k-connected, cubic, bipartite, and planar graphs, then $B\{(A_v, a_v) : v \in V(B)\}$ is also a k-connected, cubic, bipartite, and planar graph, respectively.*

Let A^1, A^2, A^3 be three disjoint graphs, $a^i \in V(A^i)$, and $N(a^i, A^i) = \{a^i_1, a^i_2, a^i_3\}$, where $i \in \{1, 2, 3\}$. Let $F = Y(A^1, a^1; A^2, a^2; A^3, a^3)$ denote the graph obtained from $(A^1 - a^1) \cup (A^2 - a^2) \cup (A^3 - a^3)$ by adding three new vertices z_1, z_2, z_3 and the set of nine new edges $\{z_j a^i_j : i, j \in \{1, 2, 3\}$ (see Fig. 2). In other words, if $B = K_{3,3}$ is the complete (X, Z)-bipartite graph with $X = \{x_1, x_2, x_3\}$ and $Z = \{z_1, z_2, z_3\}$, then F is obtained from the B by replacing each vertex x_i in X by $A^i - a^i$ so that $D(A^i - a^i, F) = \{a^i_j z_j : j \in \{1, 2, 3\}$. Let $D^i = D(A^i - a^i, F)$. If P is a Λ-packing of F, then let $P^i = P^i(F)$ be the union of components of P meeting D^i and $E^i = A^i(P) = E(P) \cap D^i$, $i \in \{1, 2, 3\}$.

If each (A^i, a^i) is a copy of the same (A, a), then we write $Y(A, a)$ instead of $Y(A^1, a^1; A^2, a^2; A^3, a^3)$.

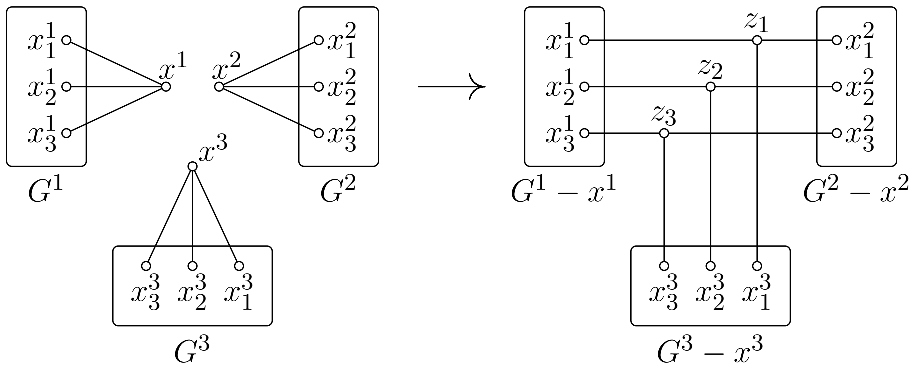

Figure 2: $Y(A^1, a^1; A^2, a^2; A^3, a^3)$

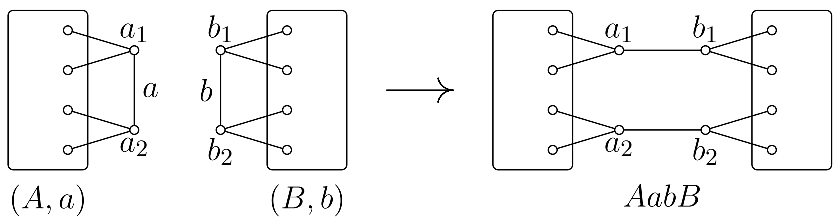

Figure 3: $AabB$

From **2.2** we have, in particular:

2.3 *Let k be an integer and $k \leq 3$. If each A^i above is a k-connected, cubic, and bipartite graph, then $Y(A^1, a^1; A^2, a^2; A^3, a^3)$ (see Fig. 2) is also a k-connected, cubic, and bipartite graph, respectively.*

Let A and B be disjoint graphs, $a = a_1a_2 \in E(A)$, and $b = b_1b_2 \in E(B)$. Let $AabB$ be the graph obtained from $(A - a) \cup (B - b)$ by adding two new edges a_1b_1, a_2b_2 (see Fig. 3). Let $Aa|bB$ be the graph obtained from $AabB$ by replacing edge a_ib_i by a 3-vertex path $a_iz_ib_i$ for each $i \in \{1, 2\}$ and by adding a new edge $z = z_1z_2$. We call z the *middle edge of $Aa|bB$* (see Fig. 4).

It is easy to see the following.

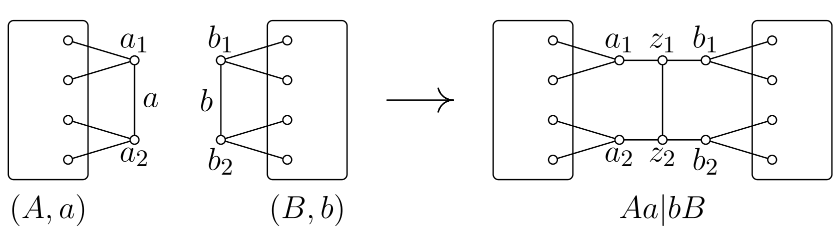

Figure 4: $Aa|bB$

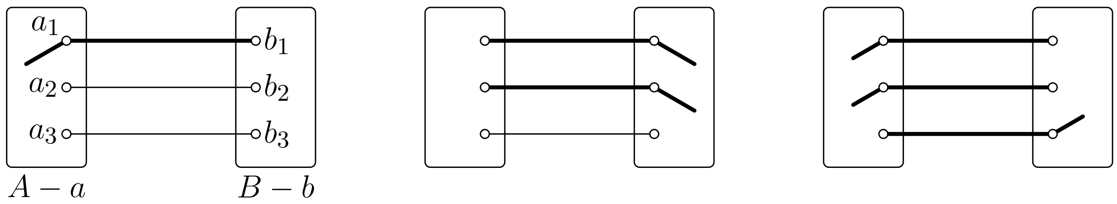

Figure 5:

2.4 *Let k be an integer and $k \le 2$. If A and B are k-connected, cubic, bipartite, and planar graphs, then both $AabB$ and $Aa|bB$ are also a k-connected, cubic, bipartite, and planar graph, respectively.*

We will use the following simple observation.

2.5 *Let A and B be disjoint graphs, $a \in V(A)$, $N(a, A) = \{a_1, a_2, a_3\}$, $b \in V(B)$, $N(b, B) = \{b_1, b_2, b_3\}$, and $G = Aa\sigma bB$, where each $\sigma(a_i) = b_i$ (see Fig. 1). Let P be a Λ-factor of G (and so $v(G) = 0 \bmod 3$) and P' be the Λ-packing of G consisting of the components (3-vertex paths) of P that meet $\{a_1b_1, a_2b_2, a_3b_3\}$.*

(a1) Suppose that $v(A) = 0 \bmod 3$, and so $v(B) = 2 \bmod 3$. Then one of the following holds (see Fig 5):

(a1.1) P' has exactly one component that has two vertices in $A - a$, that are adjacent (and, accordingly, exactly one vertex in $B - b$),

(a1.2) P' has exactly two components and each component has exactly one vertex in $A - a$ (and, accordingly, exactly two vertices in $B - b$, that are adjacent),

(a1.3) P' has exactly three components L_1, L_2, L_3 and one of them, say L_1, has exactly one vertex in $A - a$ and each of the other two L_2, L_3, has exactly two vertices in $A - a$, that are adjacent (and, accordingly, L_1 has exactly two vertices in $B - b$, that are adjacent, and each of the other two L_2, L_3, has exactly one vertex in $B - b$).

(a2) Suppose that $v(A) = 1 \bmod 3$, and so $v(B) = 1 \bmod 3$. Then one of the following holds (see Fig 6):

(a2.1) $P' = \emptyset$,

(a2.2) P' has exactly two components, say L_1, L_2, and one of the them, say L_1, has exactly one vertex in $A - a$ and exactly two vertices in $B - b$, that are adjacent, and the other component L_2 has exactly two vertices in $A - a$, that are adjacent, and exactly one vertex in $B - b$,

(a2.3) P' has exactly three components L_1, L_2, L_3 and either each L_i has exactly one vertex in $A - a$ (and, accordingly, has exactly two vertices in $B - b$, that are adjacent) or each L_i has exactly two vertices in $A - a$, that are adjacent (and, accordingly, has exactly one vertex in $B - b$).

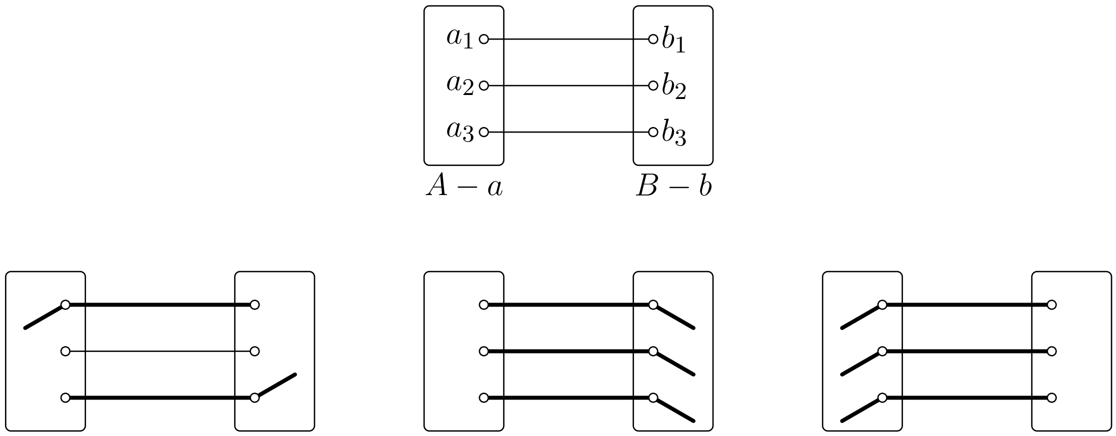

Figure 6:

3 Λ-packings in cubic 2-connected graphs

3.1 Cubic, bipartite, and 2-connected graphs

3.1 *Let $G = Y(A^1, a^1; A^2, a^2; A^3, a^3)$ (see Fig. 2) and P be a Λ-factor of G. Suppose that each A^i is a cubic graph and $v(A^i) = 0 \bmod 6$. Then $cmp(P^i) \in \{1, 2\}$ for every $i \in \{1, 2, 3\}$.*

Proof Let $i \in \{1, 2, 3\}$. Since D^i is a matching and P^i consists of the components of P meeting D^i, clearly $cmp(P^i) \le 3$. Since $v(A^i) = -1 \bmod 6$, we have $cmp(P^i) \ge 1$. It remains to show that $cmp(P^i) \le 2$. Suppose, on the contrary, that $cmp(P^1) = 3$.

Since P is a Λ-factor of G and $v(A^1 - a^1) = -1 \bmod 6$, clearly $v(P^1) \cap V(A^1 - a^1) = 5$ and we can assume (because of symmetry) that P_1 consists of three components $a^1_3 z_3 a^2_3$, $z_1 a^1_1 y^1$, and $z_2 a^1_2 u^1$ for some $y^1, u^1 \in V(A^1)$. Then $cmp(P^3) = 0$, a contradiction. $\square$

3.2 *Let A be a graph, $e = aa_1 \in E(A)$, and $G = Y(A, a)$ (see Fig. 2). Suppose that*

$(h1)$ *A is cubic,*

$(h2)$ $v(A) = 0 \bmod 6$*, and*

$(h3)$ *a has no Λ-factor containing $e = aa_1$.*

Then $v(G) = 0 \bmod 6$ and G has no Λ-factor.

Proof (uses **3.1**). Suppose, on the contrary, that G has a Λ-factor P. By definition of $G = Y(A, a)$, each A^i is a copy of A and edge $e^i = a^i a^i_1$ in A^i is a copy of edge $e = aa_1$ in A. By **3.1**, $cmp(P^i) \in \{1, 2\}$ for every $i \in \{1, 2, 3\}$. Since P is a Λ-factor of G and $v(A^i - x^i) = -1 \bmod 6$, clearly $E(P) \cap D^i$ is an edge subset of a Λ-factor of A^i for every $i \in \{1, 2, 3\}$ (we assume that edge $z_j a^i_j$ in G is edge $a^i a^i_j$ in A^i). Since $a^1 a^i_1$ belongs to no Λ-factor of A^i for every $i \in \{1, 2, 3\}$, clearly $E(P) \cap \{z_1 a^1_1, z_1 a^2_1, z_1 a^3_1\} = \emptyset$. Therefore $z_1 \notin V(P)$, and so P is not a Λ-factor of G, a contradiction. $\square$

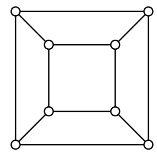

Figure 7: The cube

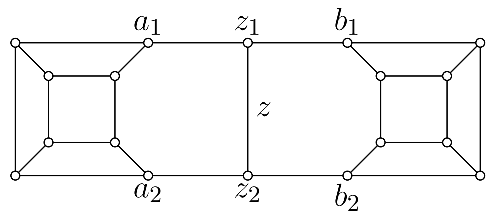

Figure 8: $K = Q_1q_1|q_2Q_2$, $v(K) = 18$

3.3 *Suppose A and B are cubic graphs and $v(A) = 2 \bmod 6$, $v(B) = 2 \bmod 6$. Let $G = Aa|bB$ with the middle edge $z = (z_1z_2)$ (see Fig. 4). Then $v(G) = 0 \bmod 6$ and G has no Λ-factor containing edge z.*

Proof Obviously $v(G) = 0 \bmod 6$. Let P be a Λ-factor of G. Let $Z = \{z_1a_1, z_2a_2\}$. Since P is a Λ-factor of G and $v(A) = 2 \bmod 6$, clearly $E(P) \cap R \neq \emptyset$.

Suppose that $|E(P) \cap Z| = 1$, say $E(P) \cap Z = z_1a_1$. Since P is a Λ-factor of G and $v(A) = 2 \bmod 6$, clearly $z_1a_1a \in P$ for some $a \in V(A)$. Then $z \notin E(P)$.

Now suppose that $|E(P) \cap Z| = 2$. Again since P is a Λ-factor of G and $v(A) = 2 \bmod 6$, clearly $a_iz_iy_i \in P$ for every $i \in \{1, 2\}$ and some $y_1, y_2 \in V(A)$, $y_1 \neq y_2$. Since $y_1 \neq y_2$ and $z = z_1z_2 \in E(G)$, clearly $z \notin E(P)$. □

A minimum simple cubic graph H with $v(H) = 2 \bmod 6$ has 8 vertices. Therefore by **3.3**, a minimum simple cubic graph M with an edge z, avoidable by every Λ-factor of M, that can be obtained by construction $Aa|bB$, has 18 vertices

The graph-skeleton Q of the cube is the only simple, cubic, and bipartite graph with 8 vertices (see Fig. 7). Moreover, Q is planar. Therefore if both A and B are disjoint copies Q_1 and Q_2 of Q, then by **3.3**, $K = Q_1q_1|q_2Q_2$ has no Λ-factor, containing the middle edge z of K, and $v(K) = 18$ (see Fig. 8). Moreover, since Q is 2-connected, bipartite, and planar, K is also 2-connected, bipartite, and planar.

3.4 *Suppose that H is a cubic graph, $v(H) = 0 \bmod 6$, $h \in E(H)$, and H has no Λ-factor containing h. Let $G = Y(H, h)$. Then $v(G) = 0 \bmod 6$ and G has no Λ-factor.*

Proof Obviously (H, h) satisfies $(h1)$, $(h2)$, and $(h3)$ of **3.2**. Therefore by **3.2**, G has no Λ-factor. □

3.5 *There are infinitely many graphs G such that G is 2-connected, cubic, and bipartite, $v(G) = 0 \bmod 6$, and G has no Λ-factor.*

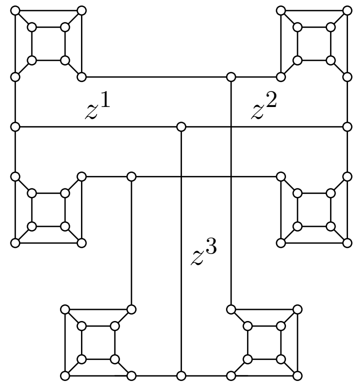

Figure 9: $R = Y(K, k)$, $v(R) = 54$

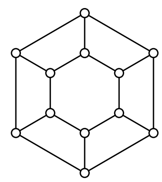

Figure 10:

Proof Follows immediately from **2.1**, **2.4**, **3.2**, **3.3**, and **3.4**. $\square$

By **3.4**, $R = Y(K, k))$ has no Λ-factor, where k is a vertex in K incident to the middle edge z of K (see Fig. 9). Obviously, R is a 2-connected, cubic, bipartite graph and $v(R) = 54$. The graph R is a smallest simple graph with these properties provided by the above construction.

3.2 Cubic, bipartite, planar, and 2-connected graphs

3.6 *Let A and B be cubic graphs, $v(A) = 0 \bmod 6$ and $v(B) = 0 \bmod 6$, $a \in V(A)$, $N(a, A) = \{a_1, a_2, a_3\}$, $b \in V(B)$, $N(b, B) = \{b_1, b_2, b_3\}$. Let $H = Aa\sigma bB$, where $\sigma(a_i) =$*

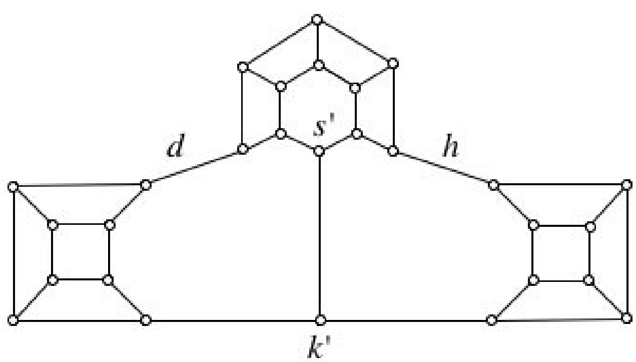

Figure 11: $H = Kk\sigma sS$, $v(H) = 28$

b_i *(see Fig. 1). Suppose that* A *has no* Λ*-factor containing* aa_1*. Then* $v(H) = 4 \bmod 6$ *and* $H - b_2$ *has no* Λ*-factor avoiding* $h = a_3b_3$*.*

Proof Let P be a Λ-factor of $H - b_2$. Since $v(A - a) = -1 \bmod 6$ and A has no Λ–factor containing aa_1, clearly $b_3a_3z \in P$ for some vertex z in A adjacent to a_3, and so $h = a_3b_3 \in E(P)$. $\square$

3.7 *Suppose that* A *and* B *are disjoint cubic graph,* $v(A) = 0 \bmod 6$*,* $v(B) = 4 \bmod 6$*,* $a = a_1a_2 \in E(A)$*,* $x \in V(B)$*,* $b = b_1b_2 \in E(B)$*,* A *has no* Λ*-factor containing* a*, and* $B - x$ *has no* Λ*-factor avoiding* b *(and so* x *is not incident to* b*). Let* $G = AabB$ *(Fig 3). Then* $v(G) = 4 \bmod 6$ *and* $G - x$ *has no* Λ*-factor.*

Proof Obviously $v(G) = 4 \bmod 6$. Suppose, on the contrary, that $G - x$ has a Λ-factor P. Since $B - x$ has no Λ-factor and $v(G) = 0 \bmod 6$, clearly $\{a_1b_1, a_2b_2\} \subseteq E(P)$ and there are vertices $a_3 \in V(A)$ and $b_3 \in V(B)$ such that (up to symmetry) $a_3a_1b_1, a_2b_2b_3 \in P$. Then $(P \cap A) \cup a_3a_1a_2$ is a Λ-factor of A containing a, a contradiction. $\square$

3.8 *Suppose that* A *and* B *are disjoint cubic graph,* $v(A) = 2 \bmod 6$*,* $v(B) = 4 \bmod 6$*,* $a = a_1a_2 \in E(A)$*,* $b = b_1b_2 \in E(B)$*, and* $B - b_1$ *has no* Λ*-factor. Let* $G = AabB$ *(Fig 3). Then* $v(G) = 0 \bmod 6$ *and* G *has no* Λ*-factor avoiding* a_2b_2*.*

Proof Suppose, on the contrary, that G has a Λ-factor P avoiding a_2b_2. Since P is a Λ-factor of G and $v(A) = 2 \bmod 6$, clearly $E(P) \cap \{a_1b_1, a_2b_2\} = a_1b_1$ and $a_3a_1b_1 \in P$ for some $a_3 \in V(A - \{a_1, a_2\}$. Then $P \cap B$ is a Λ-factor of $B - b_1$ a contradiction. $\square$

3.9 *Suppose that* A *and* B *are disjoint cubic graph,* $v(A) = 0 \bmod 6$*,* $v(B) = 0 \bmod 6$*,* $a = a_1a_2 \in E(A)$*,* $b = b_1b_2 \in E(B)$*, and* A *has no* Λ*-factor containing* a*, and* B *has no* Λ*-factor avoiding* b*. Let* $G = AabB$ *(Fig 3). Then* $v(G) = 0 \bmod 6$ *and* G *has no* Λ*-factor.*

Proof Suppose, on the contrary, that G has a Λ-factor P. Since P is a Λ-factor of G and B has no Λ-factor avoiding b, clearly $\{a_1b_1, a_2b_2\} \subseteq E(P) \cap$ and there are vertices $a_3 \in V(A)$ and $b_3 \in V(B)$ such that (up to symmetry) $a_3a_1b_1, a_2b_2b_3 \in P$. Then $(P \cap A) \cup a_3a_1a_2$ is a Λ-factor of A containing a, a contradiction. $\square$

3.10 *There are infinitely many graphs* G *such that* $v(G) = 0 \bmod 6$*,* G *is 2-connected, cubic, bipartite, and planar, and* G *has no* Λ*-factor.*

Proof Follows from **2.1**, **2.4**, **3.3**, **3.6**, **3.7**, **3.8**, and **3.9**. $\square$

Let $\mathcal{CBP}$ denote the set of cubic, bipartite, planar, and 2-connected graphs. The smallest graph G in $\mathcal{CBP}$ with $v(G) = 0 \bmod 6$ is the six-prism S, $v(S) = 12$ (see Fig. 10). Therefore by **3.6**, the smallest graph H in $\mathcal{CBP}$ that has properties, guaranteed

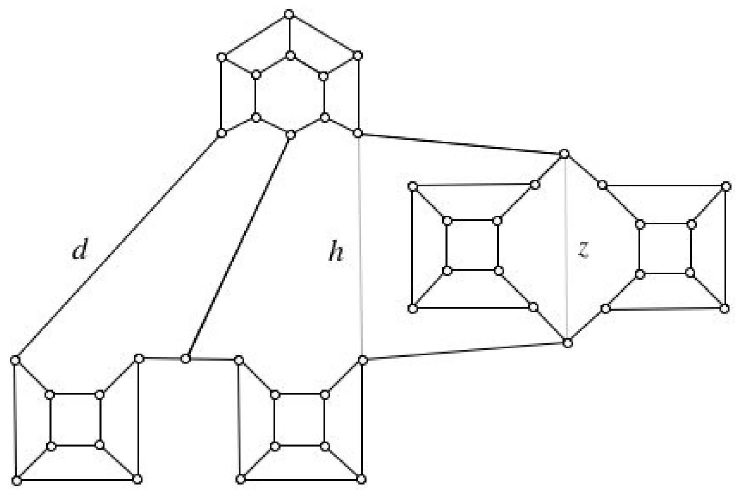

Figure 12: $D = KzhH$, $v(D) = 46$

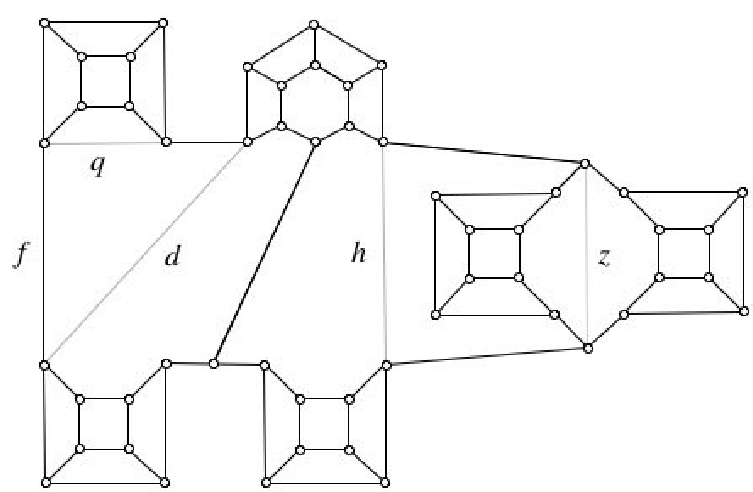

Figure 13: $F = Qqd(KzhH)$, $v(F) = 54$

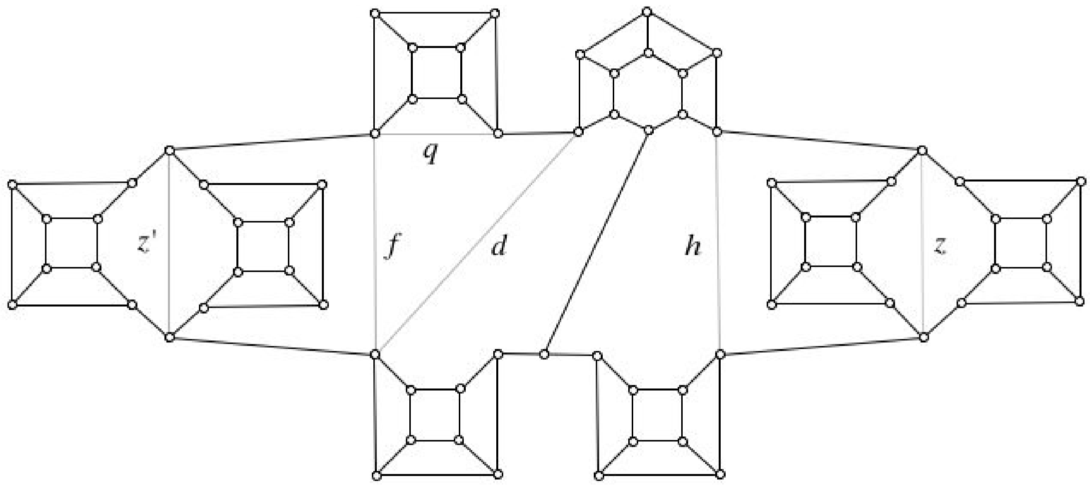

Figure 14: $N = Kz'fF$, $v(N) = 72$

by **3.6**, and that can be obtained by construction $Aa\sigma bB$ (see Fig 1), is $H = Kk\sigma sS$, $v(H) = 28$ (see Fig 11). Accordingly, by **3.7**, the smallest graph D in $\mathcal{CBP}$ that has properties, guaranteed by **3.7**, and that can be obtained by construction $AabB$ (see Fig 3), is $D = KzhH$, $v(D) = 46$ (see Fig 12). By **3.8**, the smallest graph F in $\mathcal{CBP}$ that has properties, guaranteed by **3.8**, and can be obtained by construction $AabB$, is $F = QqdD$, $v(F) = 54$ (see Fig. 13). Now by **3.9**, the smallest graph N in $\mathcal{CBP}$ with $v(N) = 0 \bmod 6$ that has no Λ-factor, and that can be obtained by the above construction, is $N = Kz'fF$, $v(N) = 72$ (see Fig. 14).

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